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# A stability criterion for steady finite amplitude convection with an external magnetic field

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In a horizontal layer of fluid, thermal expansion or the presence of dissolved salt may cause a density gradient opposite to the direction of gravity. In such cases, when the buoyancy forces are sufficient to overcome the dissipative effects, the static state becomes unstable and convective motions arise. If the layer is infinitely large in horizontal extent, the non-linear convection problem is highly degenerate, admitting many different steady-state solutions. A general necessary criterion for stability of such non-linear steady solutions is developed here for the case in which a homogeneous vertical magnetic field acts on the fluid. The criterion is demonstrated for two rigid bounding surfaces which are perfect thermal and electrical conductors, but it is applicable to more general kinds of boundary conditions.

## 1. Introduction

If a problem has several steady solutions, as is the case in convection processes, general stability criteria are quite desirable not only for the purpose of determining the preferred solution but also in order to gain knowledge about the instability mechanism itself. The idea for the present work arose from the criticism on mathematical grounds of the so-called 'relative stability' criterion of Malkus & Veronis (1958). They try to investigate the stability of a given steady state only with the aid of 'power integrals' of the stability equations and the stationary equations, by assuming that the disturbance has the form of a steady solution times a time-dependent factor. The separation assumption can be achieved by considering only infinitesimal disturbances which satisfy linear equations and which separate into a product of a space-dependent function and an exponential time function. However, the space-dependent part is then not in the form of a steady solution because the change of the form of a normalized disturbance is of the same order as the amplitude of the steady solution and the distortion of the disturbance due to interactions with the steady field is in general different from the steady field itself. Since we know that at a marginal point, i.e. zero amplitude, none of the steady solutions is preferred, the stability properties depend on the distortion of the disturbance by interaction with the steady-state field. So when Malkus & Veronis 'restrict the class of disturbances to those which have the form of steady solutions' one has to take into account the possibility that the class of disturbances considered may be empty. Nevertheless, the criterion itself, which states that only the solution with maximum heat transport, or more generally maximum amplitude, is stable, is correct if the amplitude is small enough, as will be shown in a forthcoming paper by Schlüter, Lortz & Busse. But for large amplitude motions, when amplitude expansions are no longer valid or impossible to carry out, the stability theory of convection processes is still quite unsatisfactory.

We shall consider the effects of an external magnetic field. In this case even the linear theory, treated in Chandrasekhar (1961), is rather complex. However, the advantage of the following method is that the equations do not have to be solved since only certain general properties of their solutions are dealt with. This enables us to take into account also diffusion processes through the layer which lead to additional buoyancy forces. The reader will notice that the conclusions remain almost unchanged if not only one but an arbitrary number of different kinds of materials diffuse through the layer.

## 2. The fundamental equations

Consider a horizontal fluid layer of depth $d$, on which the following forces act in the vertical direction described by the unit vector $\lambda_i$: the buoyancy force, due to gravity and changes in density; a homogeneous magnetic field of strength $H$; and any conservative force with potential $V$. Then by using the so-called Boussinesq approximation, the Navier–Stokes equations and the equation of continuity can be written (see, for instance, Chandrasekhar 1961, where the same notation is used with $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$)

$$\partial_t u_i + u_j \partial_j u_i - (\mu/4\pi\rho) H_j \partial_j H_i = \nu \Delta u_i - (\delta\rho/\rho_0) g\lambda_i - \partial_i[p/\rho + (\mu/8\pi\rho) H_j H_j + V + g x_j \lambda_j], \tag{2.1}$$

and

$$\partial_j u_j = 0. \tag{2.2}$$

The kinematic viscosity $\nu$, the magnetic permeability $\mu$, and the gravity acceleration $g$ are assumed to be constants throughout the layer.

If the fluid has constant resistivity $\eta$ then from the basic equations of hydromagnetics one can derive the equations

$$\partial_t H_i + u_j \partial_j H_i - H_j \partial_j u_i = \eta \Delta H_i, \tag{2.3}$$

and

$$\partial_j H_j = 0, \tag{2.4}$$

for the magnetic field $H_i$. In accordance with the Boussinesq approximation we have further the heat equation

$$\partial_t T + u_j \partial_j T = \kappa_T \Delta T \tag{2.5}$$

for the temperature $T$, and we treat the coefficient of thermometric conductivity $\kappa_T$ as a constant of the fluid.

If there is something dissolved with concentration $S$ in the fluid, we write the conservation equation as

$$\partial_t S + u_j \partial_j S = \kappa_S \Delta S, \tag{2.6}$$

where we have made use of (2.2). As with the other coefficients, we regard the diffusion coefficient $\kappa_S$ as a material constant.

Equations (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6) must be supplemented by an equation of state, which we approximate by

$$\rho = \rho_0[1-\alpha_T(T-T_0)+\alpha_S(S-S_0)],$$

which takes into account the fact that the fluid becomes denser on decreasing the temperature or increasing the concentration of the dissolved material.

Let us assume that the plane surfaces between which the fluid is confined are perfect conductors for heat and electric current and that they are maintained at constant temperature and constant concentration. When no motion is present, the steady solution of (2.3) and (2.4) is the externally given homogeneous field of strength $H$, and the solutions of (2.5) and (2.6) are linear functions described by the constant gradients $\beta_T$ and $\beta_S$ respectively. So for the convective state let us write

$$T-T_0 = -\beta_T x_j\lambda_j+\theta, \quad S-S_0 = \beta_S x_j \lambda_j - s, \quad H_i = H\lambda_i + h_i,$$

where $\theta$, $s$, $h_i$ are the deviations from the static state of temperature, concentration, and magnetic field, respectively.

Our fundamental system of equations takes the form

$$\left.\begin{aligned}
&\partial_t u_i + u_j\partial_j u_i - (\mu/4\pi\rho)\, h_j\partial_j h_i = \nu\Delta u_i + (\alpha_T\theta+\alpha_S s)\, g\lambda_i \\
&\qquad\qquad\qquad\qquad\qquad\qquad + (\mu/4\pi\rho)\, H\lambda_j\partial_j h_i - \partial_i\omega,\\
&\omega = p/\rho + (\mu/8\pi\rho)\, H_j H_j + V + g x_j\lambda_j + \tfrac{1}{2}(\alpha_T\beta_T+\alpha_S\beta_S)\, g x_k\lambda_k x_j\lambda_j,\\
&\partial_t h_i + u_j\partial_j h_i - h_j\partial_j u_i = \eta\Delta h_i + H\lambda_j\partial_j u_i,\\
&\partial_t\theta + u_j\partial_j\theta = \kappa_T\Delta\theta + \beta_T u_j\lambda_j,\\
&\partial_t s + u_j\partial_j s = \kappa_S\Delta s + \beta_S u_j\lambda_j,\\
&\partial_j u_j = \partial_j h_j = 0.
\end{aligned}\right\} \quad (2.7)$$

By substituting

$$u_i = (\nu/d)\, u_i', \quad t = (d^2/\nu)\, t', \quad x_i = dx_i', \quad \theta = \beta_T d\theta',$$
$$s = \beta_S ds', \quad h_i = Hh_i', \quad \omega = (\nu^2/d^2)\,\omega',$$

in (2.7) and afterwards dropping the primes, (2.7) takes the dimensionless form

$$\left.\begin{aligned}
\partial_t u_i + u_j\partial_j u_i - Mh_j\partial_j h_i &= \Delta u_i - \partial_i\omega + (R_T\theta + R_S s)\,\lambda_i + M\partial_j\lambda_j h_i,\\
M\partial_t h_i + Mu_j\partial_j h_i - Mh_j\partial_j u_i &= MP_H\Delta h_i + M\lambda_j\partial_j u_i,\\
R_T\partial_t\theta + R_T u_j\partial_j\theta &= R_T P_T\Delta\theta + R_T\lambda_j u_j,\\
R_S\partial_t s + R_S u_j\partial_j s &= R_S P_S\Delta s + R_S\lambda_j u_j,\\
\partial_j u_j = \partial_j h_j &= 0,
\end{aligned}\right\} \quad (2.8)$$

where $$M = \mu H^2 d^2/4\pi\rho, \quad R_T = \alpha_T\beta_T g d^4/\nu^2, \quad R_S = \alpha_S\beta_S g d^4/\nu^2, \quad (2.9)$$
$$P_H = \eta/\nu, \quad P_T = \kappa_T/\nu, \quad P_S = \kappa_S/\nu. \quad (2.10)$$

Let us recall the names of some of these dimensionless numbers; $P_T^{-1}R_T$ is the Rayleigh number and $P_T^{-1}$ is the Prandtl number. One of our basic assumptions is the constancy of all the material coefficients which occur, although in real fluids these coefficients depend on the temperature, the concentration, and perhaps on the magnetic field. The external parameters of the convective state are described only by the numbers (2.9), and from their form we see that if these numbers are fixed and the depth $d$ is large enough, the differences of temperature and concentration and the magnetic field will be small enough for the material coefficients to be sufficiently constant.

## 3. The boundary conditions

If we are concerned with rigid bounding-surfaces such as metal, we have to require that the velocity vector $u_i$ must vanish at the surface because viscosity prevents the fluid from slipping. Since $T$ and $S$ are fixed at the surface, $\theta$ and $s$ have to vanish there. The requirement that the adjoining medium be a perfect conductor leads to the boundary condition that the vertical component of the magnetic field $h_i$ and the vertical derivative of the vertical component of the current density have to vanish. So our full set of boundary conditions is

$$u_i = 0, \quad \theta = 0, \quad s = 0, \quad \lambda_j h_j = 0, \quad \partial_j \lambda_j \lambda_i \epsilon_{ikl} \partial_k h_l = 0, \tag{3.1}$$

at the planes between which the fluid is confined. In horizontal directions we assume the layer to be infinite, and we require all occurring functions to be bounded.

Let us now look at the signs of the dimensionless numbers (2.9) and (2.10). $M$, $P_H$, $P_T$, and $P_S$ are always positive while $R_T$ and $R_S$ may have either sign. But if $R_T$ and $R_S$ both have negative signs it would mean that the buoyancy force is negative, and it is physically obvious that in such a case no convection is possible. This can also be seen mathematically. Multiply the first of (2.8) by $u_i$ (forming the scalar product), the second by $h_i$, the third by $-\theta$, the fourth by $-s$, add the results, and average over the whole layer; then

$$\begin{aligned}\tfrac{1}{2}\partial_t[(u_i u_i)_m + M(h_i h_i)_m - R_T(\theta^2)_m - R_S(s^2)_m] = (u_i \Delta u_i)_m + MP_H(h_i \Delta h_i)_m \\ - R_T P_T(\theta\Delta\theta)_m - R_S P_S(s\Delta s)_m,\end{aligned} \tag{3.2}$$

where $(\ )_m$ denotes the average. The other terms cancel or can be transformed into surface integrals which vanish in consequence of the boundary conditions (3.1). Using Green's theorem and the boundary conditions, the right-hand side of (3.2) can be transformed into

$$-([\partial_j u_i]\,\partial_j u_i)_m - MP_H([\partial_j h_i]\,\partial_j h_i)_m + R_T P_T([\partial_j \theta]\,\partial_j \theta)_m + R_S P_S([\partial_j s]\,\partial_j s)_m,$$

which is negative definite if neither $R_T$ nor $R_S$ is positive. So as time goes on the positive definite expression on the left-hand side of (3.2) tends to zero for any initial condition. This means that the static state is asymptotically stable.

## 4. Reduction of the vector equations to scalar equations

It is quite clear that the non-linear system (2.8), together with the boundary conditions (3.1), is a problem of considerable complexity. Yet the whole system has certain symmetries, which we shall now discuss.

Let us write the general integrals of the continuity equation in a form which is appropriate to the geometry of the present problem. We introduce the differential operators

$$\delta_i \equiv \partial_j \lambda_j \partial_i - \Delta\lambda_i, \quad \gamma_i \equiv \epsilon_{ijk}\partial_j\lambda_k. \tag{4.1}$$

Then

$$u_i = \delta_i v_1 + \gamma_i v_2 \tag{4.2}$$

is a solution of (2.2) for arbitrary differentiable functions $v_1$, $v_2$, which we may call potentials of the velocity. The analogous form to (4.2) for the magnetic field is

$$h_i = \delta_i g_1 + \gamma_i g_2. \tag{4.3}$$

By operating on the first equation of (2.8) with $\delta_i$ and then with $\gamma_i$, we find

$$\partial_t \Delta_2 \Delta v_1 + \delta_i(u_j \partial_j u_i - M h_j \partial_j h_i) = \Delta_2 \Delta^2 v_1 - \Delta_2(R_T\theta + R_S s) + M\partial_j\lambda_j \Delta_2 \Delta g_1,$$

$$\partial_t \Delta_2 v_2 + \gamma_i(u_j \partial_j u_i - M h_j \partial_j h_i) = \Delta_2 \Delta v_2 + M\partial_j \lambda_j \Delta_2 g_2,$$

where

$$\Delta_2 \equiv \Delta - \partial_j\lambda_j\partial_k\lambda_k$$

is the two-dimensional Laplacian operator in the horizontal plane. Operating on the second vector equation of (2.8) with $-\delta_i$ and then with $-\gamma_i$ yields

$$-M\partial_t \Delta_2 \Delta g_1 - M\delta_i(u_j\partial_j h_i - h_j \partial_j u_i) = -MP_H \Delta_2 \Delta^2 g_1 - M\partial_j\lambda_j\Delta_2\Delta v_1,$$

$$-M\partial_t \Delta_2 g_2 - M\gamma_i(u_j\partial_j h_i - h_j \partial_i u_i) = -MP_H \Delta_2 \Delta g_2 - M\partial_j\lambda_j \Delta_2 v_2.$$

Finally, after substituting (4.2) and (4.3) into the non-linear terms, we have six scalar equations for the set of six scalar variables $v_1$, $v_2$, $g_1$, $g_2$, $\theta$, $s$, which we abbreviate by $v$. The boundary conditions for them can be found with the aid of equations (2.8), (3.1), (4.2) and (4.3). They are

$$v_1 = \partial_j\lambda_j v_1 = v_2 = g_1 = \partial_j\lambda_j\partial_k\lambda_k g_1 = \partial_j\lambda_j g_2 = \theta = s = 0 \tag{4.4}$$

on the bounding surface of the fluid.

To shorten the writing work we introduce the matrix differential operators

$$V \equiv \begin{pmatrix} \Delta_2\Delta^2 & 0 & M\partial_j\lambda_j\Delta_2\Delta & 0 & -R_T\Delta_2 & -R_S\Delta_2 \\ 0 & \Delta_2\Delta & 0 & M\Delta_2\partial_j\lambda_j & 0 & 0 \\ -M\partial_j\lambda_j\Delta_2\Delta & 0 & -MP_H\Delta_2\Delta^2 & 0 & 0 & 0 \\ 0 & -M\Delta_2\partial_j\lambda_j & 0 & -MP_H\Delta_2\Delta & 0 & 0 \\ -R_T\Delta_2 & 0 & 0 & 0 & R_TP_T\Delta & 0 \\ -R_S\Delta_2 & 0 & 0 & 0 & 0 & R_SP_S\Delta \end{pmatrix},$$

$$U \equiv \begin{pmatrix} \Delta_2\Delta & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -M\Delta_2\Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & -M\Delta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_T & 0 \\ 0 & 0 & 0 & 0 & 0 & R_S \end{pmatrix}$$

and a vector valued quadratic differential operator by

$$Q(v',v) \equiv \begin{pmatrix} \delta_i[(\delta_j v_1'+\gamma_j v_2')\,\partial_j(\delta_i v_1+\gamma_i v_2)-M(\delta_j g_1'+\gamma_j g_2')\,\partial_j(\delta_i g_1+\gamma_i g_2)] \\ \gamma_i[(\delta_j v_1'+\gamma_j v_2')\,\partial_j(\delta_i v_1+\gamma_i v_2)-M(\delta_j g_1'+\gamma_j g_2')\,\partial_j(\delta_i g_1+\gamma_i g_2)] \\ -M\delta_i[(\delta_j v_1'+\gamma_j v_2')\,\partial_j(\delta_i g_1+\gamma_i g_2)-(\delta_j g_1'+\gamma_j g_2')\,\partial_j(\delta_i v_1+\gamma_i v_2)] \\ -M\gamma_i[(\delta_j v_1'+\gamma_j v_2')\,\partial_j(\delta_i g_1+\gamma_i g_2)-(\delta_j g_1'+\gamma_j g_2')\,\partial_j(\delta_i v_1+\gamma_i v_2)] \\ R_T(\delta_j v_1'+\gamma_j v_2')\,\partial_j\theta \\ R_S(\delta_j v_1'+\gamma_j v_2')\,\partial_j s \end{pmatrix}.$$

Then our system of six scalar equations can be written

$$\partial_t Uv+Q(v,v) = Vv. \tag{4.5}$$

The operators contain the dimensionless numbers (2.9), which describe the external physical parameters, and the numbers (2.10), which describe the properties of the fluid. For the discussion of how the amplitude of $v$ depends on the various parameters, we shall change $R_T$ and hold all other numbers fixed, though the problem (4.5) is completely symmetric in $R_T$ and $R_S$, i.e. in $\theta$ and $s$.

Steady solutions satisfy the equation

$$Q(v,v) = Vv. \tag{4.6}$$

To test their stability we superimpose small disturbances $\tilde{v}$ on to the steady functions $v$, and after linearizing we derive from (4.5) the stability equation

$$\sigma U\tilde{v}+Q(v,\tilde{v})+Q(\tilde{v},v) = V\tilde{v}, \tag{4.7}$$

where we have made an exponential ansatz for the time-dependence by setting

$$\partial_t\tilde{v} = \sigma\tilde{v}.$$

We regard the stability equations (4.7) together with the boundary conditions (4.4) as an eigenvalue problem for the growth rate $\sigma$. The stability problem is then whether or not equation (4.7) has positive eigenvalues $\sigma$ for a given steady-state $v$.

## 5. The properties of the linear system

After neglecting the quadratic terms in (4.5), i.e. considering $v$ to be of infinitesimal amplitude, we can write with exponential time-dependence

$$\sigma Uv = Vv, \tag{5.1}$$

which is identical with (4.7) if we neglect the interaction terms between the disturbance and the steady motion.

Defining a scalar product

$$\langle v',v\rangle \equiv (v_1'v_1)_m+(v_2'v_2)_m+(g_1'g_1)_m+(g_2'g_2)_m+(\theta'\theta)_m+(s's)_m,$$

for functions $v$ and $v'$ which satisfy the boundary conditions (4.4), we find that the operators $U$ and $V$ have the following self-adjoint property

$$\langle v', Vv\rangle = \langle v, Vv'\rangle; \quad \langle v', Uv\rangle = \langle v, Uv'\rangle\dagger. \tag{5.2}$$

† This means the eigenvalue problem (5.1) is the Euler–Lagrange equation of the variational principle $\delta[\sigma\langle v, Uv\rangle-\langle v, Vv\rangle] = 0$.

In spite of these relations the eigenvalue $\sigma$ in the problem (5.1) will in general be complex and so will the eigenfunctions. If $v$ is an eigenfunction of (5.1) and $v^*$ its complex conjugate, then

$$\sigma\langle v^*, Uv\rangle = \langle v^*, Vv\rangle,$$

or with the aid of (5.2)

$$\sigma[\langle v, Uv^*\rangle + \langle v^*, Uv\rangle] = \langle v^*, Vv\rangle + \langle v, Vv^*\rangle.$$

Since the operators $V$ and $U$ are real, the right-hand side of the last equation and the factor of $\sigma$ are real. So

$$\langle v^*, Uv\rangle = 0 \tag{5.3}$$

if $\sigma$ is complex and hence also

$$\langle v^*, Vv\rangle = 0. \tag{5.4}$$

Equations (5.3) and (5.4) are possible because, unlike the ordinary convection problem, $U$ and $V$ are in general not definite.

Nevertheless, we are interested only in real eigenvalues of (5.1) and in particular in the question as to which value $R_T = R_T^{(0)}$ has to be chosen for $\sigma = 0$. That means we are looking for a value $R_T^{(0)}$ at which stationary convection sets in, and furthermore for the lowest of those values. We write

$$V^{(0)}v^{(0)} = 0, \tag{5.5}$$

where the superscript on $V^{(0)}$ means we have replaced $R_T$ by $R_T^{(0)}$, and we now consider $R_T^{(0)}$ as an eigenvalue of the problem (5.5).

We shall first show that all eigenvalues $R_T^{(0)}$ are real. If $R_T^{(0)}$ and hence $v^{(0)}$ were complex, then

$$\langle v^{(0)*}, V^{(0)}v^{(0)}\rangle = 0,$$

and with the definition of $V^{(0)}$

$$R_T^{(0)}\langle v^{(0)*}, Wv^{(0)}\rangle + \langle v^{(0)*}, V_1 v^{(0)}\rangle = 0, \tag{5.6}$$

where

$$V_1 = V^{(0)} - R_T^{(0)}W, \quad W = \begin{pmatrix} 0 & 0 & 0 & 0 & -\Delta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\Delta_2 & 0 & 0 & 0 & P_T\Delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$R_T^{(0)}W$ is the matrix which contains the fifth row and fifth column of $V^{(0)}$ in the respective places and has zeros everywhere else. So $V_1$ has the same property of self-adjointness as $V^{(0)}$ (equation (5.2)), and hence the second term of (5.6) is real. The coefficient of $R_T^{(0)}$, which is also real, is

$$\langle v^{(0)*}, Wv^{(0)}\rangle = -(\theta^{(0)*}\Delta_2 v_1^{(0)})_m + P_T(\theta^{(0)*}\Delta\theta^{(0)})_m - (v_1^{(0)*}\Delta_2\theta^{(0)})_m.$$

If $R_T^{(0)} = 0$, our statement is right. So let us consider the case $R_T^{(0)} \neq 0$; then

$$-\Delta_2 v_1^{(0)} + P_T\Delta\theta^{(0)} = 0,$$

and hence

$$\langle v^{(0)*}, Wv^{(0)}\rangle = -(v_1^{(0)*}\Delta_2\theta^{(0)})_m = -P_T(\theta^{(0)*}\Delta\theta^{(0)})_m = P_T([\partial_j\theta^{(0)*}]\,\partial_j\theta^{(0)})_m, \tag{5.7}$$

which is positive unless $\theta^{(0)} \equiv 0$. The case $\theta^{(0)} \equiv 0$ leads to the uninteresting case $v^{(0)} \equiv 0$. So the only possibility is that the eigenvalue $R_T^{(0)}$ is always real.

A further necessary condition that the convection first sets in as a stationary motion is that the growth rate $\sigma$ is negative for values $R_T$ which are smaller than $R_T^{(0)}$ because otherwise $R_T^{(0)}$ would not be the lowest critical value. If we consider in (5.1) the dependence of $\sigma$ and $v$ on $R_T$ we have to require $\partial\sigma/\partial R_T \geqslant 0$ for $R_T = R_T^{(0)}$.

By differentiating (5.1) with respect to $R_T$ at $R_T = R_T^{(0)}$ we find

$$(\partial\sigma/\partial R_T)\, U^{(0)} v^{(0)} = V^{(0)}\, \partial v/\partial T_T + W v^{(0)} \tag{5.8}$$

in addition to equation (5.1).

Equation (5.8) is an inhomogeneous equation for $\partial v/\partial R_T$, and since we may anticipate that (5.1) has solutions for all values $R_T$, (5.8) has solutions as well. Then it is necessary that the inhomogeneous part be orthogonal to the solution of the homogeneous part of (5.8)

$$(\partial\sigma/\partial R_T)\,\langle v^{(0)}, U^{(0)} v^{(0)}\rangle = \langle v^{(0)}, W v^{(0)}\rangle.$$

The right-hand side is positive as we saw in (5.7). So if the physical parameters, described by the dimensionless numbers (2·9), (2.10) are such that the motion sets in as stationary convection we have

$$\langle v^{(0)}, U^{(0)} v^{(0)}\rangle > 0, \tag{5.9}$$

which we shall need later when we discuss the stability of the non-linear steady solutions.

Equations (5·1) and (5·5) admit solutions which separate the vertical and horizontal dependence and which satisfy the equation

$$\Delta_2 v + a^2 v = 0 \tag{5.10}$$

in horizontal planes.

For every value of the overall wave-number $a$, (5.5) is an eigenvalue problem of ordinary differential equations. By looking at the matrix operator $V$ we see that the variables $v_2$ and $g_2$ can be separated from the rest of the variables. Furthermore, $v_2 \equiv g_2 \equiv 0$ is the only solution (see Chandrasekhar 1961). Without solving the remaining problem we mention some properties of its solutions. There exists a lowest eigenvalue $R_T^{(0)}$ which is not degenerate. The components $v_1$, $\theta$, and $s$ of the corresponding eigenfunction are even with respect to the middle of the layer while $g_1$ is odd. The neutral curve $R_T^{(0)}\,(a^2)$ attains its minimum for a finite value $a_c^2$.

## 6. Non-linear steady solutions and their stability

Solutions of the non-linear equation (4.6) can be approximated by a formal expansion

$$v = \epsilon v^{(0)} + \epsilon^2 v^{(1)} + \epsilon^3 v^{(2)} + \dots, \tag{6.1}$$

when the amplitude $\epsilon$ is small but finite and the parameter $R_T$ does not deviate much from its marginal value $R_T^{(0)}$, so that we can also write

$$R_T = R_T^{(0)} + \epsilon R^{(1)} + \epsilon^2 R^{(2)} + \dots. \tag{6.2}$$

Since $R_T$ is an externally given parameter, we regard (6.2) as the definition of $\epsilon$ because we shall see that the values $R^{(\nu)}$ are determined by certain existence conditions.

Substituting (6.1) and (6.2) into the stability equations we can apply the usual perturbation theory

$$\sigma = \sigma^{(0)} + \epsilon\sigma^{(1)} + \epsilon^2\sigma^{(2)} + \dots,$$

$$\tilde{v} = \tilde{v}^{(0)} + \epsilon\tilde{v}^{(1)} + \epsilon^2\tilde{v}^{(2)} + \dots.$$

Thus we see that for the lowest order in $\epsilon$ the stability equations reduce to (5.1) with $R_T = R_T^{(0)}$ and $\sigma = \sigma^{(0)}$ and the stationary equations reduce to (5.5).

If we now restrict the class of disturbances to those which have the same overall wave-number $a$ as the steady solutions, then we know that (5.1) has no positive eigenvalues $\sigma^{(0)}$ and that the highest eigenvalue is zero. For $\sigma^{(0)} = 0$ and the same wave-number the equations (5.1) and (5.5) become identical. The vertical dependence of the disturbances is the same as for the steady state while the horizontal dependence may be different. So let us write

$$v^{(0)} = fw, \quad \tilde{v}^{(0)} = f\tilde{w}, \tag{6.3}$$

where $f$ is a column matrix whose elements depend only on the vertical coordinate $x_j\lambda_j$. The functions $w$ and $\tilde{w}$ satisfy the wave equation (5.10), whose bounded solution we may write in the form

$$\left.\begin{aligned} w &= \sum_{\substack{m=-N\\ m\neq 0}}^{+N} C_m w_m, \quad \tilde{w} = \sum_m \tilde{C}_m \tilde{w}_m, \\ w_m &= \exp(i\mathbf{k}_m.\mathbf{r}), \quad \tilde{w}_m = \exp(i\tilde{\mathbf{k}}_m.\mathbf{r}), \end{aligned}\right\} \tag{6.4}$$

where $\mathbf{k}_m$, $\tilde{\mathbf{k}}_m$ are two-dimensional wave vectors with

$$|\mathbf{k}_m|^2 = |\tilde{\mathbf{k}}_m|^2 = a^2,$$

and $\mathbf{r}$ is the position vector in the horizontal plane. In order that the steady solution in (6.4) be real, we require that for each $\mathbf{k}_m$ there is a $\mathbf{k}_{-m} = -\mathbf{k}_m$ with $C_{-m} = C_m^*$. Note that no summation convention is used for sums over the particular solutions $w_m$. We choose the normalization

$$\sum_{m=-N}^{+N} |C_m|^2 = 1 \tag{6.5}$$

for the steady state.

At second-order in $\epsilon$ the stationary system is

$$-R^{(1)}Wv^{(0)} + Q(v^{(0)}, v^{(0)}) = V^{(0)}v^{(1)}. \tag{6.6}$$

This is an inhomogeneous equation for $v^{(1)}$, whose homogeneous part has nontrivial solutions. Forming the scalar product with equation (6·6) and any such solution $v^{(0)\prime}$ and using the self-adjoint property of $V^{(0)}$, we find

$$-R^{(1)}\langle v^{(0)\prime}, Wv^{(0)}\rangle + \langle v^{(0)\prime}, Q(v^{(0)}, v^{(0)})\rangle = \langle v^{(0)\prime}, V^{(0)}v^{(1)}\rangle = \langle v^{(1)}, V^{(0)}v^{(0)\prime}\rangle = 0.$$

This is a condition which can be satisfied by choosing a special value $R^{(1)}$.

With the explicit expression for $Q$ and the symmetry of the functions $v^{(0)}$ we see that each component of $Q(v^{(0)}, v^{(0)})$ has the opposite vertical symmetry to the respective component of $v^{(0)}$. Hence

$$\langle v^{(0)\prime}, Q(v^{(0)}, v^{(0)})\rangle = 0.$$

From equation (5.7) we know that $\langle v^{(0)\prime}, Wv^{(0)}\rangle \neq 0$ if we choose $v^{(0)\prime} = v^{(0)}$. Hence

$$R^{(1)} = 0.$$

Then the analogous stability equation becomes

$$\sigma^{(1)} U^{(0)} \tilde{v}^{(0)} + Q(\tilde{v}^{(0)}, v^{(0)}) + Q(v^{(0)}, \tilde{v}^{(0)}) = V^{(0)} \tilde{v}^{(1)}, \tag{6.7}$$

with the existence condition

$$\sigma^{(1)}\langle v^{(0)\prime}, U^{(0)} \tilde{v}^{(0)}\rangle + \langle v^{(0)\prime}, Q(\tilde{v}^{(0)}, v^{(0)}) + Q(v^{(0)}, \tilde{v}^{(0)})\rangle = 0.$$

The second term is found to be zero by the same argument as above. Then choosing $v^{(0)\prime} = \tilde{v}^{(0)}$ we know from equation (5.9) that the coefficient of $\sigma^{(1)}$ is unequal to zero and hence $\sigma^{(1)}$ has to be zero.

At this place we should discuss the case of other boundary conditions. For the conclusion $R^{(1)} = \sigma^{(1)} = 0$ the symmetry property of the linear functions has been used. But for unsymmetric boundary conditions which are different at the top and bottom of the layer, for instance if the lower surface is rigid and the upper surface free, the symmetry argument no longer holds. Nevertheless, without a magnetic field $R^{(1)}$ and $\sigma^{(1)}$ vanish even for unsymmetric boundary conditions, as will be shown by Schlüter *et al.* (1965), while magnetic effects can yield contributions to $R^{(1)}$ and $\sigma^{(1)}$ in the unsymmetric case. The respective integrals are trilinear in the linear solutions and hence the horizontal average is unequal to zero only if the linear steady-state solutions have regular hexagonal structure such that their **k**-vectors add to zero. This is the formal reason why in such a case the regular hexagonal cell-pattern has a different stability behaviour. A similar asymmetry argument holds if one considers temperature dependence of the material coefficients such as viscosity; see Busse (1962), Palm (1960) and Segel (1965).

In order to find differences in the behaviour of various steady solutions we have to go to higher approximations. Before doing so let us discuss some features of the solutions of (6.6) and (6.7) for $R^{(1)} = \sigma^{(1)} = 0$.

When we substitute the linear solutions into the inhomogeneous terms of (6.6) and use $v_2^{(0)} \equiv g_2^{(0)} \equiv 0$, we obtain for the fourth component apart from the factor $-M$

$$\gamma_i[(\delta_j v_1^{(0)})\, \partial_j \delta_i g_1^{(0)} - (\delta_j g_1^{(0)})\, \partial_j \delta_i v_1^{(0)}]. \tag{6.8}$$

Further

$$\gamma_i(\delta_j v_1^{(0)})\, \partial_j \delta_i g_1^{(0)} = \epsilon_{ikl}\lambda_l[\partial_k(\partial_m \lambda_m \partial_j - \Delta\lambda_j)\, v_1^{(0)}]\, \partial_j(\partial_n \lambda_n \partial_i - \Delta\lambda_i)\, g_1^{(0)} = 0.$$

That the latter expression vanishes can be verified by using the fact that $v_1^{(0)}$ and $g_1^{(0)}$ are components of (6.3) and hence have the same horizontal dependence $w$ with $\Delta_2 w + a^2 w = 0$. In the same way we conclude that

$$\gamma_i(\delta_j g_1^{(0)})\, \partial_j \delta_i v_1^{(0)} = 0.$$

Thus the expression (6.8), i.e. the fourth component of $Q(v^{(0)}, v^{(0)})$, vanishes. The second component of $Q(v^{(0)}, v^{(0)})$, which contains terms quite analogous to those in (6.8), is zero too. Since the matrix operator $V$ separates $v_2$ and $g_2$ from the rest of the variables, there is a solution of (6·6) with

$$v_2^{(1)} \equiv g_2^{(1)} \equiv 0. \tag{6.9}$$

Considering the analogous equation (6.7) for the disturbances one finds that the second and fourth components of $Q(\tilde{v}^{(0)}, v^{(0)})$ cancel with the respective components of $Q(v^{(0)}, \tilde{v}^{(0)})$ Thus

$$\tilde{v}_2^{(1)} \equiv \tilde{g}_2^{(1)} \equiv 0 \tag{6.10}$$

as well. The rest of the inhomogeneous terms of (6.6) have the form

$$\sum_{kl} G(\mathbf{k}_k . \mathbf{k}_l, x_j \lambda_j)\, w_k w_l C_k C_l.$$

So (6.6) has a solution of the form

$$v^{(1)} = \sum_{kl} F(\phi_{kl}, x_j \lambda_j)\, C_k C_l w_k w_l, \tag{6.11}$$

where $F$ is a column matrix and

$$\phi_{kl} \equiv a^{-2}\mathbf{k}_k . \mathbf{k}_l.$$

The analogous form for the solutions of the equations (6.7) is

$$\begin{aligned} \tilde{v}^{(1)} &= \sum_{kl} F(\phi_{kl}, x_j \lambda_j)\,(C_k \tilde{C}_l w_k \tilde{w}_l + \tilde{C}_k C_l \tilde{w}_k w_l) \\ &= 2\sum_{kl} F(\phi_{kl}, x_j \lambda_j)\, C_k \tilde{C}_l w_k \tilde{w}_l. \end{aligned} \tag{6.12}$$

We now consider the next order of $\epsilon$ in the equations (4.6) and (4.7), expecting to find a splitting up of the eigenvalues $\sigma$.

$$\left.\begin{aligned} &-R^{(2)} W v^{(0)} + Q(v^{(1)}, v^{(0)}) + Q(v^{(0)}, v^{(1)}) = V^{(0)} v^{(2)}, \\ &\sigma^{(2)} U^{(0)} \tilde{v}^{(0)} - R^{(2)} W \tilde{v}^{(0)} + Q(v^{(1)}, \tilde{v}^{(0)}) + Q(v^{(0)}, \tilde{v}^{(1)}) + Q(\tilde{v}^{(1)}, v^{(0)}) \\ &\qquad\qquad + Q(\tilde{v}^{(0)}, v^{(1)}) = V^{(0)} \tilde{v}^{(2)}. \end{aligned}\right\} \tag{6.13}$$

The existence conditions are that the scalar product of the left-hand sides of (6.13) with any solution $v^{(0)\prime}$ of the homogeneous part of (6.13) vanishes.

$$\left.\begin{aligned} &-R^{(2)}\langle v^{(0)\prime}, W v^{(0)}\rangle + \langle v^{(0)\prime}, Q(v^{(1)}, v^{(0)}) + Q(v^{(0)}, v^{(1)})\rangle = 0, \\ &\sigma^{(2)}\langle v^{(0)\prime}, U^{(0)} \tilde{v}^{(0)}\rangle - R^{(2)}\langle v^{(0)\prime}, W \tilde{v}^{(0)}\rangle + \langle v^{(0)\prime}, Q(v^{(1)}, \tilde{v}^{(0)}) \\ &\qquad + Q(v^{(0)}, \tilde{v}^{(1)}) + Q(\tilde{v}^{(1)}, v^{(0)}) + Q(\tilde{v}^{(0)}, v^{(1)})\rangle = 0. \end{aligned}\right\} \tag{6.14}$$

Substituting the representations (6.3), (6.4), and (6.11), (6.12) into the inhomogeneous terms of (6.13) and using the relations (6.9) and (6.10) we see that the first, third, fifth, and sixth components of $Q$ are trilinear forms in $C_k w_k$, whose coefficients depend only on the vertical co-ordinate and the three possible scalar products between the three occurring $\mathbf{k}$-vectors. Taking $v^{(0)\prime} = f w_n^*$ as the fundamental solution of the homogeneous part of (6.13) and recalling that the second and fourth components of $f$ are zero, the first of the existence conditions (6.14) can be written

$$O = -R^{(2)} \sum_{m=-N}^{+N} C_m N_1 \overline{w_n^* w_m} + \sum_{k,l,m=-N}^{+N} \begin{pmatrix} kl \\ km \\ lm \end{pmatrix} C_k C_l C_m \overline{w_n^* w_k w_l w_m}, \tag{6.15}$$

where we have integrated over the vertical co-ordinate. The over-bar means a horizontal average. $N_1$ is a constant which with the aid of (5.7) is known to be positive. The parenthesized symbol in the equation (6.15) is an abbreviation for a function $L(\phi_{kl}, \phi_{km}, \phi_{lm})$. In the following, both notations will be used for the same function. Since $\phi_{-m,n} = -\phi_{mn}$, it should be clear what a minus sign in an argument of the parenthesized symbol means. Since the expressions in (6.11) are symmetric in $k$ and $l$, we can assume that the function $L(\phi_{kl}, \phi_{km}, \phi_{lm})$ is symmetric in the second and third variables.

We now restrict the disturbances to those which have the same **k**-vectors as the steady-state functions, i.e. $\tilde{w}_m = w_m$. Then the coefficients in the representation (6.12) are also symmetric in $k$ and $l$, and we can write the second existence condition (6.14) analogously to the first.

$$0 = \sum_{m=-N}^{+N} [\sigma^{(2)} \tilde{C}_m N_2 \overline{w_n^* w_m} - R^{(2)} \tilde{C}_m N_1 \overline{w_n^* w_m}] + \sum_{k,l,m=-N}^{+N} \begin{pmatrix} kl \\ km \\ lm \end{pmatrix} [C_k C_l \tilde{C}_m + 2C_m C_k \tilde{C}_l] \overline{w_n^* w_k w_l w_m}. \quad (6.16)$$

The constant $N_2$ is positive, when convection sets in as a stationary motion (cf. equation (5.9)). The horizontal average in (6.15) and (6.16) is zero unless in each term the sum of the **k**-vectors is zero. So the terms with two **k**-vectors contribute only if $m = n$, while those with four **k**-vectors contribute only if

(i) $k = n, l = -m$;
or (ii) $l = n, k = -m,\ m \neq -n$;
or (iii) $m = n,\ k = -l, l \neq n, l \neq -n$.

Thus (6.15) yields

$$\begin{aligned}
R^{(2)} N_1 C_n &= \sum_{m=-N}^{+N} \begin{pmatrix} -nm \\ nm \\ -mm \end{pmatrix} C_n |C_m|^2 + \sum_{\substack{m=-N \\ \neq -n}}^{+N} \begin{pmatrix} -nm \\ -mm \\ nm \end{pmatrix} C_n |C_m|^2 + \sum_{\substack{l=-N \\ \neq n \\ \neq -n}}^{+N} \begin{pmatrix} -ll \\ -nl \\ nl \end{pmatrix} C_n |C_l|^2 \\
&= C_n \sum_{m=-N}^{+N} \left[ \begin{pmatrix} -nm \\ nm \\ -mm \end{pmatrix} + \begin{pmatrix} -nm \\ -mm \\ nm \end{pmatrix} + \begin{pmatrix} -mm \\ -nm \\ nm \end{pmatrix} \right] |C_m|^2 \\
&\qquad - C_n \left[ \begin{pmatrix} nn \\ -nn \\ -nn \end{pmatrix} + \begin{pmatrix} -nn \\ -nn \\ nn \end{pmatrix} + \begin{pmatrix} -nn \\ nn \\ -nn \end{pmatrix} \right] |C_n|^2 \\
&= C_n \sum_{m=1}^{N} \left[ \begin{pmatrix} -nm \\ nm \\ -mm \end{pmatrix} + \begin{pmatrix} nm \\ -nm \\ -mm \end{pmatrix} + \begin{pmatrix} -nm \\ -mm \\ nm \end{pmatrix} + \begin{pmatrix} nm \\ -mm \\ -nm \end{pmatrix} + \begin{pmatrix} -mm \\ -nm \\ nm \end{pmatrix} \right. \\
&\qquad \left. + \begin{pmatrix} -mm \\ nm \\ -nm \end{pmatrix} \right] |C_m|^2 - C_n \left[ \begin{pmatrix} nn \\ -nn \\ -nn \end{pmatrix} + 2 \begin{pmatrix} nn \\ -nn \\ nn \end{pmatrix} \right] |C_n|^2. \quad (6.17)
\end{aligned}$$

After dividing by $C_n$ the $n$th equation is identical with the $(-n)$th equation $(n = 1, \ldots, N)$.

From (6.16) it is found that

$$R^{(2)} N_1 \tilde{C}_n = \sigma^{(2)} N_2 \tilde{C}_n + \sum_m \left[\begin{pmatrix} -nm \\ nm \\ -mm \end{pmatrix} + 2\begin{pmatrix} nm \\ -nm \\ -mm \end{pmatrix}\right] C_n C_m^* \tilde{C}_m$$

$$+ \sum_{m \neq -n} \left[\begin{pmatrix} -nm \\ -mm \\ nm \end{pmatrix} + 2\begin{pmatrix} -mm \\ -nm \\ nm \end{pmatrix}\right] C_n C_m^* \tilde{C}_m + \sum_{\substack{l \neq n \\ \neq -n}} \left[\begin{pmatrix} -ll \\ -nl \\ nl \end{pmatrix} + 2\begin{pmatrix} -nl \\ -ll \\ nl \end{pmatrix}\right] |C_l|^2 \tilde{C}_n.$$

The left side of the latter equation cancels with the last term of the right side apart from the last term of (6.17). Then using the symmetry in the second and third variables of $L(\phi_{kl}, \phi_{km}, \phi_{lm})$ equations (6.15) and (6.16) take their final form

$$R^{(2)} N_1 = \sum_{m=1}^{N} T_{nm} |C_m|^2 \quad (n = 1, \ldots, N); \tag{6.18}$$

$$0 = \sigma^{(2)} N_2 \tilde{C}_n + \sum_{m=-N}^{+N} T_{nm} C_n C_m^* \tilde{C}_m \quad (n = -N, \ldots, -1, 1, \ldots, N); \tag{6.19}$$

$$T_{nm} \equiv \begin{cases} \begin{pmatrix} nn \\ -nn \\ -nn \end{pmatrix} + 2\begin{pmatrix} -nn \\ nn \\ -nn \end{pmatrix} & \text{for} \quad m = \pm n, \\[2ex] 2\left[\begin{pmatrix} -nm \\ nm \\ -mm \end{pmatrix} + \begin{pmatrix} nm \\ -nm \\ -mm \end{pmatrix} + \begin{pmatrix} -mm \\ -nm \\ nm \end{pmatrix}\right], & \text{otherwise.} \end{cases}$$

From the definition of the function $L(\phi_{kl}, \phi_{km}, \phi_{lm})$ the following properties of the matrix $T$ can be derived:

$$\left.\begin{aligned} T_{nm} &= T_{-n,m} = T_{n,-m} = T_{-n,-m}, \\ T_{nm} &= T_{mn}. \end{aligned}\right\} \tag{6.20}$$

Note further that for $m = n$ or $m = -n$ the elements of $T$ are equal to each other. Since $T$ is symmetric, $T_{nm} C_m^* C_n$ is Hermitian and hence all eigenvalues $\sigma^{(2)}$ in (6.19) are real. The equations (6.19) have non-trivial solutions $\tilde{C}_m$ if and only if the characteristic equation

$$\det(N_2 \sigma^{(2)} \delta_{nm} + T_{nm} C_m^* C_n) = 0 \quad \text{or} \quad \det(N_2 \sigma^{(2)} \delta_{nm} / |C_n|^2 + T_{nm}) = 0 \tag{6.21}$$

is satisfied. The question whether or not the equation (6.21) has positive roots can be decided with a simpler equation in which the positive factor $N_2/|C_n|^2$ is omitted. We can see this, for instance, by forming Rayleigh's quotient. So the stability conclusions are unchanged if we consider the equation

$$\det\,[2\sigma^{(2)} \delta_{nm} + T_{nm}] = 0 \quad (n, m = -N, \ldots, -1, 1, \ldots, N)$$

instead of (6.21). After subtracting the $m$th column from the $(-m)$th column in the foregoing determinant and adding the $(-n)$th row to the $n$th row we find with the aid of (6.20) that $N$ eigenvalues $\sigma^{(2)}$ are zero. The rest of the eigenvalues satisfy the characteristic equation

$$\det[\sigma^{(2)}\delta_{nm}+T_{nm}] = 0 \quad (n, m = 1, \dots, N). \tag{6.22}$$

The equation (6.22) and the equation (6.18) together with the normalization condition (6.5) are the basic relations for the following discussion. Since the linear and first-order problems were not explicitly solved, the elements of the matrix $T$ are unknown. But $T$ has the symmetry property and its diagonal elements are equal to each other. This is sufficient for deriving a general stability criterion.

Equations (6.18), together with the normalization condition

$$\sum_{m=1}^{N} |C_m|^2 = \tfrac{1}{2},$$

represent an inhomogeneous system of $(N+1)$ linear equations which determine the $(N+1)$ values $R^{(2)}, |C_1|^2, \dots, |C_N|^2$. This means: not every linear solution is an approximate to a steady non-linear solution with arbitrarily small amplitude, because the coefficients in the linear superposition (6.4) are determined by the non-linearities apart from an arbitrary phase constant. Furthermore, the directions of the **k**-vectors have to be chosen such that the solutions $|C_m|^2$ of (6.18) are positive. This is a constraint on the matrix $T$.

Letting

$$T_{nm} = dA_{nm}+B_{nm},$$

where $d$ is the diagonal element of $T$ and all elements of $A$ are equal to one, then $B$ is a symmetric matrix whose diagonal elements vanish. Equations (6.18) and (6.22) thus become

$$b = \sum_{m=1}^{N} B_{nm}|C_m|^2, \quad b \equiv N_1 R^{(2)} - \tfrac{1}{2}d, \tag{6.23}$$

$$\det(\sigma^{(2)}\,\delta_{nm}+dA_{nm}+B_{nm}) = 0, \tag{6.24}$$

respectively.

It should be realized that the constant $b$ is zero for the two-dimensional motion, i.e. when $N = 1$. So a solution with positive $b$ has a larger value of $R^{(2)}$ and hence less amplitude than the two-dimensional solution.

We are now going to show that if $b$ is positive then (6.24) necessarily has positive roots. If $d$ is negative, i.e. the value $R^{(2)}$ for the two-dimensional solution is negative, then the coefficient of $(\sigma^{(2)})^{N-1}$ is negative. So in this case, according to the sign rules of Descartes, there are positive roots of (6.24) for all values of $b$, which would mean there is no stable solution at all. But the method described is appropriate only if the motion begins with arbitrarily small amplitude. There may also be non-linear solutions which have finite amplitude at the critical point, for instance in cases with negative $R^{(2)}$.

Now let us consider the case $d > 0$. The solution of (6.23) may be written

$$|C_n|^2 = b\sum_{m}(B^{-1})_{nm}. \tag{6.25}$$

The absolute term of the polynomial (6.24) is

$$\det(dA+B) = \det(dB^{-1}AB+B) = \det(B)\det(dB^{-1}A+\delta),$$

where $\delta$ is the unit matrix. With the aid of (6.25), we obtain

$$(B^{-1}A)_{nm} = \sum_{k=1}^{N} (B^{-1})_{nk} = \frac{1}{b}|C_n|^2 \quad (n, m = 1, \ldots, N).$$

Hence

$$\det(dB^{-1}A+\delta) = \begin{vmatrix} 1+\dfrac{d|C_1|^2}{b} & -1 & -1 & -1 & \ldots & -1 & -1 \\ \dfrac{d|C_2|^2}{b} & 1 & 0 & 0 & \ldots & 0 & 0 \\ \dfrac{d|C_3|^2}{b} & 0 & 1 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \dfrac{d|C_n|^2}{b} & 0 & 0 & 0 & \ldots & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+\dfrac{d}{b}\sum_k |C_k|^2 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \dfrac{d|C_2|^2}{b} & 1 & 0 & 0 & \ldots & 0 & 0 \\ \dfrac{d|C_3|^2}{b} & 0 & 1 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \dfrac{d|C_n|^2}{b} & 0 & 0 & 0 & \ldots & 0 & 1 \end{vmatrix} = 1+\frac{d}{b}\sum_m |C_m|^2 > 0.$$

Thus if $\det B$ is negative, our statement is proved, because the absolute term is negative in that case. So let us next consider the case $\det B > 0$. By subtracting any row in (6.24) from all others we first notice that all coefficients of the polynomial are linear in $d$, because all principal minors are. In particular (6.24) can be written in the form

$$(\sigma^{(2)})^N - [\tfrac{1}{2}\sum_{nm} B_{nm}^2](\sigma^{(2)})^{N-2} + \ldots + \det B + d[N(\sigma^{(2)})^{N-1} + \ldots$$
$$+ \det B(1/b)\sum_m |C_m|^2] = 0 \quad (N > 2). \qquad (6.26)$$

According to the sign rule of Descartes there are at least two positive roots for $\det B > 0$ if $d$ is zero, because all roots are real and the coefficients have at least two sign changes. Now suppose there exists a positive $d$ such that all zeros of (6.26) are negative, i.e. no coefficient of the polynomial is negative. Then there must be a value $d_0$ such that at least one coefficient, say that of

$$(\sigma^{(2)})^\nu, \quad 0 < \nu < N-1,$$

is equal to zero, while none of the others is negative. Since the constant term is positive for all $d$, all zeros are negative for $d = d_0$.

On the other hand, since all zeros are real, the zeros of the derivative of (6.26) with respect to $\sigma^{(2)}$ are located between the zeros of (6.26). In particular all zeros of the derivative are negative and so are the zeros of all higher derivatives. But the $\nu$th derivative has a vanishing constant term and hence has a vanishing root.

This is a contradiction. Thus (6.26) has at least two positive roots for all positive $d$ when $\det B$ is positive.

The remaining case $\det B = 0$ does not occur for $b > 0$, because for $\det B = 0$ the equations (6.23) for $|C_m|^2$ have only solutions if

$$\sum_{m=1}^{N} b|C_m|^2 = 0,$$

which is not possible for positive $b$.

Next we show that for a solution with negative $R^{(2)}$, positive $\sigma^{(2)}$-values exist. If no root $\sigma^{(2)}$ were positive, then $T_{nm}$ in equation (6.22) would be a positive semi-definite matrix. In particular

$$0 \leqslant \sum_{nm} |C_n|^2 T_{nm} |C_m|^2 = \sum_{n} |C_n|^2 N_1 R^{(2)},$$

and this is obviously a contradiction for $R^{(2)} < 0$, $N_1 > 0$.

So we can formulate the stability criterion. A steady solution with small but finite amplitude is unstable unless

$$0 \leqslant N_1 R^{(2)} \leqslant \tfrac{1}{2}d.$$

A measure for the amplitude is for instance the average density of the kinetic energy or the horizontal average of the vertical heat transport. The case of negative $R^{(2)}$ means that there is a solution for $R_T$ smaller than $R_T^{(0)}$ because $\epsilon^2$ in (6.2) is positive. Such a solution may be called subcritical. Since $\frac{1}{2}d$ is the value of $N_1 R^{(2)}$ for the two-dimensional solution the final form of the stability criterion is: *A steady solution with small but finite amplitude is unstable if it is subcritical or if it has smaller amplitude than the two-dimensional solution.*

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